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Closure Categories

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In this work I present a condensed, but self-contained review of the categorical formulation of order structures, and the induced equivalence of the categories of closure spaces and complete atomistic lattices.

1. INTRODUCTION

Since their introduction in analysis by E. H. Moore and F. Riesz (E. H. Moore, 1909; Riesz, 1909) closure operators have found applications in many areas, such as logic (Hertz, 1922; Tarski, 1929) and topology (Kuratowski, 1922; Čech, 1937). Of particular interest in a closure space are the fixed points of the closure operator, for example, the deductive closure of a set of axioms or the closed subsets of a topological space. Now it turns out that the collection of these fixed points forms a complete lattice with respect to the inclusion order, whose greatest lower bound is just the intersection (Baer, 1959; Everett, 1944; Monteiro and Ribeiro, 1942; Ore, 1943a,b; Riguet, 1948; Ward, 1942). One is then led to a range of equivalences between particular classes of closure operators and particular types of lattices.

In this work I shall present a survey of the categorical structures underlying these equivalences. There are several motivations for an abstract synthesis of such reasonably well known results. First of all, it provides especially transparent examples of many standard categorical notions which, to the uninitiated, often seem rather obscure in their full generality. Second, it exhibits in an interesting way the structural features which lie at the origin of properties such as the meet-closedness of the fixed point lattice of a closure operator. Finally, and most importantly, the categorical constructions to be

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presented in the following find a direct application to many specific problems of physical and mathematical interest.

For example, any physical system can be represented by its property lattice, a complete atomistic orthocomplemented lattice, or by its state space, a set together with an orthogonality relation. Explicitly, we have that $a < b$ if b is actual whenever a is actual; in other words, the partial order is physical implication with respect to actuality. On the other hand, we have that $\mathscr{E}_1 \perp \mathscr{E}_2$ if there exists a definite experimental project which is certain for $\mathscr{E}_1$ and impossible for $\mathscr{E}_2$; in other words, two states are orthogonal if they can be sharply distinguished. Further, the descriptions of a system in terms of its state space and property lattice are physically dual: to each property we can associate the set of states for which it is actual; and to each state we can associate the set of all of its actual properties. The standard axioms then assure that these correspondences are faithful: states correspond exactly to atomic properties; properties correspond exactly to biorthogonal sets of states (Aerts, 1982; Piron, 1990). The resulting categorical equivalence between the two representations then allows a compact characterisation of notions such as classical variables (Moore, 1995), observables (Piron, 1976, §2.4), and maximal deterministic evolutions (Faure *et al.*, 1995).

On the other hand, a matroid is defined to be a simple algebraic closure operator satisfying an exchange condition. A special case of some importance is that of projective geometries, exactly matroids whose associated fixed point lattice is modular (Faure and Frölicher, 1996). Here a detailed categorical study has led to a deep generalization of the fundamental representation theorem to cover general semilinear maps and Hermitian forms (Faure and Frölicher, 1993, 1994, 1995). First, fixing a hyperplane H of the Arguesian projective geometry G, one can embed G as a hyperplane in the projective geometry $\overline{G}$ of endomorphisms of G with axis H. Then, defining $V = \overline{G} \backslash G$ and fixing a point $0 \in V$, one can prove that V is a vector space over a division ring whose multiplicative group is the set of isomorphisms of $\overline{G}$ with center 0, and that G is isomorphic to the set of rays of V. Second, one can prove that any nondegenerate morphism f: $G_1 \backslash K_1 \to G_2$ can be extended to a morphism $\bar{f}$: $\overline{G}_1 \backslash K_1 \to \overline{G}_2$, which restricts to a semilinear map $\bar{A}$: $V_1 \to V_2$ with respect to the division ring homomorphism s defined by $\bar{f} \circ \lambda = s(\lambda) \circ h$. Finally, an orthogonality relation $\perp$ on the projective geometry G induces a morphism $\bar{h}$: $G \to G^*$ into the dual geometry, whose corresponding semilinear map f: $V \to V^*$ defines a definite Hermitian form on V.

The rest of this work is organized as follows. In Section 2, I provide the necessary definitions of category theory; for more details see, for example, Adámek *et al.* (1990), Borceux (1994) or Mac Lane (1971). In Section 3, I discuss the categorical formulation of general order structures. Finally, in Section 4, I introduce closure operators and their fixed point lattices, before defining equivalent categories of complete atomistic lattices and closure spaces in Section 5.

2. CATEGORY THEORY

In terms of its application to the concrete structures of physics, category theory is perhaps best considered as a hierarchy of object-structure relations, with morphisms as relations between objects, functors as relations between morphisms, and natural transformations as relations between functors. This hierarchy can in fact be formalized by considering (a) the quasicategory of categories, whose objects are categories and whose morphisms are functors between them, and (b) functor quasicategories, whose objects are functors between two fixed categories and whose morphisms are natural transformations between them. More importantly, however, the conception of a hierarchy of object-structure relations lies at the heart of the very definitions of morphisms, functors, and natural transformations, the imposed conditions being nothing more than unicity requirements on induced relations.

First, let Ob be some class of objects whose structural relations are of interest to us, and suppose that for A, $B \in$ Ob the structural relations from A to B can be collected into a set Hom(A, B). Clearly, identification provides a relation from A to itself, and so we are led to require the existence of an identity morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$. Further, if f relates the objects A and B, and g relates the objects B and C, it is natural to suppose that there is an induced relation $g \circ f$ of A to C. We are therefore led to require the existence of a composition law for morphisms. Now the intuitive notion of relation is indifferent to the order in which we concatenate relations. We are then led to postulate that composition be associative. Next, for the intuitive notion of identification to be coherent, the relation induced by some morphism f and an identification should be f itself. We are then led to postulate that the identity morphisms are compositional units. Finally, any structural relation involves a unique ordered pair of objects, namely the domain and codomain. We are then led to require that the Hom-sets be pairwise disjoint. From this point of view, a category is then a quadruple (Ob, Hom, id, $\circ$) consisting of:

(1) a class Ob of objects;
(2) for each ordered pair (A, B) of objects a set Hom(A, B) of morphisms;
(3) for each object A a morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$;
(4) a composition law associating to each pair of morphisms $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$ a morphism $g \circ f \in \mathrm{Hom}(A, C)$;

which is such that:

(1) $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, and $h \in \mathrm{Hom}(C, D)$;
(2) $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$ for all $f \in \mathrm{Hom}(A, B)$;
(3) the sets $\mathrm{Hom}(A, B)$ are pairwise disjoint.

Next, a relation between morphisms should respect the structurally important features involved in the concept of morphism. Hence a functor F should relate the domain and codomain objects of the initial morphism to those of the final morphism. Further, the identity morphisms form a distinguished class, and so should be preserved by any functor. Finally, given two morphisms $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$ there are two natural induced relations from FB and FB, namely $F(g \circ f)$ and $Fg \circ Ff$. Since an induced relation should be uniquely specified, we are led to postulate that functors preserve composition. From this point of view, a functor from the category X to the category Y is then a family of maps F which associates to each object A in X an object FA in Y, and to each morphism $f \in \mathrm{Hom}(A, B)$ a morphism $Ff \in \mathrm{Hom}(FA, FB)$, which is such that:

(1) $F\,\mathrm{id}_A = \mathrm{id}_{FA}$ for all $A \in \mathrm{Ob}$;
(2) $F(g \circ f) = Fg \circ Ff$ all $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B,C)$.

Finally, let F and G be functors from the category X to the category Y. Any relation of F to G should induce a relation θ_A between the objects FA and GA for each object $A \in \mathrm{Ob}\,(X)$. Further, given a morphism $f \in \mathrm{Hom}(A, B)$ in X there are two natural relations from FA to GB, namely $\theta_B \circ Ff$ and $Gf \circ \theta_A$. Once again, since an induced relation should be unique, we are led to postulate that the two be equal. From this point of view, a natural transformation from the functor F to the functor G is then a map θ which assigns to each object A of X a morphism $\theta_A \in \mathrm{Hom}(FA, GA)$ in Y, such that for each $f \in \mathrm{Hom}(A, B)$ in X we have that

$$\theta_B \circ Ff = Gf \circ \theta_A$$

It can be argued that the methodological utility of category theory lies in its unified treatment of universal constructions. These can often be performed at either the local level in terms of morphisms and limits, or the global level in terms of functors and adjunctions. For example, the intuitive notion of a product can be formalized either locally as an object together with a family of projections, or globally as an adjoint of the diagonal functor. Explicitly, a diagram in the category X is a functor ∇ from J to X, where J is a small category, that is a category with a set of objects. A natural source of the diagram ∇ is then an object X in X together with a family of morphisms

$p_j \in \mathrm{Hom}(X, \nabla(j))$, which respect the structural constraints encoded in the index category J:

$$p_j = \nabla(t) \circ p_i \qquad \forall t \in \mathrm{Hom}(i, j)$$

A limit of the diagram ∇ is a natural source $p_i \in \mathrm{Hom}(X, \nabla(j))$ such that for any natural source $\bar{p}_j \in \mathrm{Hom}(\bar{X}, \nabla(j))$ there exists a unique morphism $\bar{f} \in \mathrm{Hom}(\bar{X}, X)$ such that

$$\bar{p}_j = p_j \circ f$$

On the other hand, let F be a functor from X to Y, and G be a functor from Y to X. Then F is called a left adjoint of G and G is called a right adjoint of F, written $F \dashv G$, if there exist natural transformations

$$\eta\colon \quad \mathrm{Id}_x \to G \circ F, \qquad \epsilon\colon \quad F \circ G \to \mathrm{Id}_Y$$

such that

$$\epsilon F \circ F\eta = \mathrm{id}F, \qquad G\epsilon \circ \eta G = \mathrm{id}G$$

Note that in this case F preserves colimits, and G preserves limits. Finally, if η and ϵ are natural isomorphisms, then F and G are said to define an equivalence.

A useful notion in category theory, which will play a central role in the following, is that of a monad on the category X, that is, a triple (T, η, μ) consisting of (Godement, 1957):

(1) a functor T from X to X;
(2) a natural transformation η from Id_x to T;
(3) a natural transformation μ from $T \circ T$ to T;

which is such that

$$\mu \circ T\mu = \mu \circ \mu T, \qquad \mu \circ T\eta = \mathrm{id}T, \qquad \mu \circ \eta T = \mathrm{id}T$$

Monads are in fact closely related to adjunctions. Indeed, let L be a functor from X to Y and R be a functor from Y to X, such that $L \dashv R$ via the natural transformations η and ϵ. Then one can prove that $(R \circ L, \eta, R\epsilon L)$ is a monad on X (Huber, 1961). Further, as we shall see next, each monad arises in this way.

First, let (T, η, μ) be a monad on the category X. A T-algebra on X is a pair (A, α) consisting of an object $A \in \mathrm{Ob}(X)$ and a morphism $\alpha \in \mathrm{Hom}(TA, A)$, which is such that

$$\alpha \circ \eta_A = \mathrm{id}_A, \qquad \alpha \circ T\alpha = \alpha \circ \mu_A$$

Let (A, α) and (B, β) be T-algebras. A T-morphism from (A, α) to (B, β) is a morphism $f \in \mathrm{Hom}(A, B)$, which is such that

$$f \circ \alpha = \beta \circ Tf$$

Then T-algebras together with T-morphisms and the original composition law form a category X^T, called the Eilenberg–Moore category associated to the monad (T, η, μ) (Eilenberg and Moore, 1965). For example, the Eilenberg–Moore category associated to the word monad on *Set* is the category of monoids, and the Eilenberg–Moore category associated to the power monad on *Set* is the category of complete join semilattices. Let us define the families of maps

$$U^T\colon \quad (A, \alpha) \mapsto A; f \mapsto f, \qquad F^T\colon \quad A \mapsto (TA, \mu_A); f \mapsto Tf$$

Then U^T is a functor from X^T to X, and F^T is a functor from X to X^T. We have that $U^T \circ F^T = T$ and $F^T \vdash U^T$. Finally, for a monad of the form $T = R \circ L$ there then exists a unique functor K from Y to $X^{R \circ L}$ such that

$$R = U^{R \circ L} \circ K, \qquad F^{R \circ L} = K \circ L$$

Explicitly,

$$K\colon \quad A \mapsto (RA, \mathrm{R}\epsilon_A); \qquad f \mapsto Rf$$

On the other hand, let (T, η, μ) be a monad on the category X and define:

(1) $\mathrm{Ob}(X_T) = \mathrm{Ob}(X)$;
(2) $\mathrm{Hom}_T(A, B) = \mathrm{Hom}(A, TB)$;
(3) $id_A = \eta_A$;
(4) $g \circ f = \mu_C \circ T\, g \circ f$.

Then X_T is a category, called the Kleisli category associated to the monad (T, η, μ) (Kleisli, 1965). For example, the Kleisli categories of monads on the category of sets and functions can be identified with algebraic theories, that is, theories involving operations on sets constrained by axioms expressed as equalities. For a detailed treatment of the relationships between monads, theories, and topoi see Barr and Wells (1985). Let us define the families of maps

$$U_T\colon \quad A \mapsto TA; \qquad f \mapsto \mu_B \circ Tf, \qquad F_T\colon \quad A \mapsto A; \qquad f \mapsto \eta_B \circ f$$

Then U_T is a functor from X_T to X, and F_T is a functor from X to X_T. We have that $U_T \circ F_T = T$ and $F_T \vdash U_T$. Finally, for a monad of the form $T = R \circ L$ there exists a unique functor J from $X_{R \circ L}$ to Y such that

$$L = J \circ F_{R \circ L}, \qquad U_{R \circ L} = R \circ J$$

Explicitly,

$$J\colon\ A \mapsto LA; \qquad f \mapsto \epsilon_{LB} \circ Lf$$

3. ORDER AND CATEGORY

As already pointed out by Eilenberg and Mac Lane (1945) in their foundational paper on category theory, one of the simplest examples of a category is a thin category, that is, a category in which each Hom-set contains at most one element. As we shall see in this section, such categories are nothing more than preordered classes, and functors between them are just isotone maps. The notions of product and coproduct then correspond to the greatest lower bound and least upper bound, respectively. This leads naturally to the notion of a lattice, and in particular to the identification of complete lattices with complete thin categories. A consideration of adjunctions in such categories then illustrates the importance of join (meet)-preserving maps. Indeed, $F \dashv G$ if and only if F preserves the join, G preserves the meet, and the two are related by the standard Galois duality.

A preorder on the class X is a relation $<$ which is reflexive and transitive:

(1) $a < a$ for each $a \in X$.
(2) If $a < b$ and $b < c$, then $a < c$.

Recall that a thin category is a category X such that for each ordered pair of objects (a, b) there exists at most one morphism in $\mathrm{Hom}(a, b)$.

Lemma 3.1. Preordered classes are in bijective correspondence with thin categories.

Indeed, let us define

$$\mathrm{Hom}(a, b) = \begin{cases} \{(a, b)\}: & a < b \\ \emptyset; & \text{otherwise} \end{cases}$$

Then the symmetry of $<$ guarantees the existence of identity morphisms, namely $\mathrm{id}_a = (a, a)$, and the transitivity of $<$ provides a uniquely defined associative composition law, namely $(b, c) \circ (a, b) = (a, c)$. Note that for any thin category $(X, <)$ the opposite category $(X, <)^{\mathrm{op}}$, defined by $\mathrm{Ob}(X^{\mathrm{op}}) = \mathrm{Ob}(X)$ and $\mathrm{Hom}^{\mathrm{op}}(a, b) = \mathrm{Hom}(a, b) = \mathrm{Hom}(b, a)$ with $f * g = g \circ f$, is also thin and so a preordered class. Explicitly, $a <^{\mathrm{op}} b$ if and only if there exists a morphism $f \in \mathrm{Hom}^{\mathrm{op}}(a, b)$, which is the case if and only if $f \in \mathrm{Hom}(b, a)$, that is, if and only if $b < a$.

Recall that a functor from $(X_1, <_1)$ to $(X_2, <_2)$ is a family of maps which associates to each object $a \in X_1$ an object $Fa \in x_2$, and to each morphism $f\colon a \to b$ a morphism $Ff\colon Fa \to Fb$, such that $F\mathrm{id}_a = \mathrm{id}_{Fa}$ and $F(g \circ f) = Fg \circ Ff$.

Lemma 3.2. Functors between the preordered sets $(X_1, <_1)$ and $(X_2, <_2)$ considered as thin categories are in bijective correspondence with isotone maps $F: X_1 \to X_2$.

Indeed, if $a_1 <_1 b_1$, then $Fa_1 <_2 Fb_1$, since there exists a morphism $f: a_1 \to b_1$, and so a morphism Ff: $\mathrm{Fa}_1 \to Fb_1$.

Recall that a natural transformation $\theta: F \to G$ is a map which associates to each object $a_1 \in X_1$ a morphism $\theta_{a_1}: Fa_1 \to Ga_1$ such that for each morphism $f: a_1 \to b_1$ we have that $\theta_{b_1} \circ Ff = Gf \circ \theta_{a_1}$.

Lemma 3.3. There exists a natural transformation $\theta: F \to G$ if and only if $Fa_1 <_2 Ga_1$ for each $a_1 \in X_1$.

Indeed, $Fa_1 <_2 Ga_1$, since θ_{a_1} is a morphism from Fa_1 to Ga_1. Note that, by the thinness of the category, θ is uniquely defined if it exists.

Recall that a product is an object a together with a family of morphisms $p_j: a \to a_j$ such that for any object $\bar{a}$ and any family of morphisms $\bar{p}_j: \bar{a} \to a_j$, there exists a unique morphism $f: \bar{a} \to a$ such that $\bar{p}_j = p_j \circ f$, and that a coproduct is a product in the opposite category.

Lemma 3.4. Let $(X, <)$ be a preordered class considered as a thin category, and $\mathcal{A} \subseteq X$. We have the following results:

(1) A product $\Pi\mathcal{A}$, if it exists, is a greatest lower bound of $\mathcal{A}$.
(2) A coproduct $\Pi\mathcal{A}$, if it exists, is a least upper bound of $\mathcal{A}$.

Indeed, translating the definition of a product in terms of the preorder, a is a product if and only if $a < a_j$ for each $a_j \in \mathcal{A}$, and for any $\bar{a}_j < a_j$ we have that $\bar{a} < a$. The corresponding result for coproducts then follows by duality.

A partial order on the set X is an antisymmetric preorder $<$:

(1) $a < a$ for each $a \in X$.
(2) If $a < b$ and $b < c$, then $a < c$.
(3) If $a < b$ and $b < a$, then $a = b$.

A pair $(X, <)$, where X is a set and $<$ is a partial order on X, is called a poset. A poset is then nothing more than a small thin category for which no two distinct objects are isomorphic.

There are two reasons for restricting our attention to small categories at this stage. The first is convenience: by so doing, we obtain the usual category of posets with isotone maps as a subcategory of the category of small categories. For example, the unicity of natural transformations for posets implies that the functor category $\mathbf{F}((X_1, <_1), (X_2, <_2))$ is also a poset. In particular, by taking functor categories as power objects, we then recover the well-known Cartesian closedness of the category of posets. The second motivation has more content. In many cases we start with a preordered collection which is not a set, but whose induced poset of equivalence classes is. A case in point is the collection of possible definite experimental projects which could eventually be performed on a given particular physical system. Being in no way delimited, this collection does not admit a reasonable mathematical characterization. By way of contrast, it is reasonable to suppose that the induced collection of properties is well circumscribed and so a set.

A poset $(X, <)$ is called a lattice if for all $a, b \in X$ there exists a greatest lower bound $a \wedge b$ and a least upper bound $a \vee b$, that is, if finite products and coproducts exist. Note that if $(X, <)$ is a lattice, then so is $(X, <)^{\mathrm{op}}$. Indeed, since passing to the opposite category interchanges limits and colimits, we have that $a \wedge^{\mathrm{op}} b = a \vee b$ and $a \vee^{\mathrm{op}} b = a \wedge b$. Further, the category of lattices is in fact an algebraic construct, since lattice orders are equationally definable. Explicitly, posing

(1) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a;$

(2) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c;$

(3) $a \wedge (a \vee b) = a \vee (a \wedge b) = a;$

we have that $a < b$ if and only if $a \wedge b = a$ if and only if $a \vee b = b$.

A poset $(X, <)$ is called a complete meet semilattice if the greatest lower bound $\wedge\mathcal{A}$ of an arbitrary subset $\mathcal{A}$ of X exists; or a complete join semilattice if the least upper bound $\vee\mathcal{A}$ of an arbitrary subset $\mathcal{A}$ of X exists. In fact a poset $(X, <)$ is a complete meet lattice if and only if it is a complete join lattice, with

$$\vee\mathcal{A} = \wedge\{b \in X | (\forall a \in \mathcal{A})\, a < b\}, \qquad \wedge\mathcal{A} = \vee\{b \in X | (\forall a \in \mathcal{A})\, b < a\}$$

Recall that a category is called complete if all limits exist.

Lemma 3.5. Let $(X, <)$ be a poset. Then $(X, <)$ is a complete lattice if and only if it is complete as a thin category.

Indeed, the naturality condition on a natural source $p_j \in \mathrm{Hom}(X, \nabla(j))$, namely that $p_j = \nabla(t) \circ p_i$ for each index morphism $t \in \mathrm{Hom}(i, j)$, is redundant by the thinness of the category. Hence limits reduce to products of index objects. A poset is then complete as a thin category if and only if it has all products, that is, if and only if it is a complete lattice. Note that the MacNeille completion of a poset, originally defined by Dedekind cuts (MacNeille, 1937), is itself categorical, being the injective hull (Banaschewski and Bruns, 1967). Further, such initial completions can be defined, although not necessarily realized, for any construct (Herrlich, 1976): for example, the category of preordered sets is the MacNeille completion of the category of posets (Alderton, 1985, 1986).

Recall that $F \dashv G$, that is, F is a left adjoint of G, if and only if there exist natural transformations η: $\mathrm{Id}_{x_1} \to G \circ F$ and ϵ: $F \circ G \to \mathrm{Id}_{x_2}$ such that $\epsilon F \circ F\eta = \mathrm{id}\, F$ and $G\epsilon \circ \eta G = \mathrm{id}\, G$.

Lemma 3.6. Let $(X_1, <_1)$ and $(X_2, <_2)$ be posets, and F: $X_1 \to X_2$ and G: $X_2 \to X_1$ be isotone maps. Then $F \dashv G$ if and only if $a_1 < GFa_1$ for each $a_1 \in X_1$, and $FGa_2 < a_2$ for each $a_2 \in X_2$.

Indeed, the second condition is nothing more than the requirement that there exist the required natural transformations η: $\mathrm{Id}_{x_1} \to G \circ F$ and ϵ: $F \circ G \to \mathrm{Id}_{x_2}$. Note that this condition is equivalent to the well-known Galois duality $a_1 < Ga_2$ if and only if $Fa_1 < a_2$.

Recall that in any category, if $F \dashv G$, then F preserves colimits and G preserves limits. Translating in terms of the partial order, we then have that F preserves the meet and G preserves the join. Further, given solution set conditions which turn out to be trivial for thin categories, one can compute adjunctions via the adjoint functor theorems. Explicitly:

Lemma 3.7. Let $(X_1, <_1)$ and $(X_2, <_2)$ be complete lattices, F: $X_1 \to X_2$ preserve the least upper bound, and G: $X_2 \to X_1$ preserve the greatest lower bound. Then $F \dashv G$ if and only if

$$Ga_2 = \vee\{x_1 \in X_1 | Fx_1 < a_2\}, \qquad Fa_1 = \wedge\{x_2 \in X_2 | a_1 < Gx_2\}$$

Note that in any category, if $F_1 \dashv G_1$ and $F_2 \dashv G_2$, then $F_2 \circ F_1 \dashv G_1 \circ G_2$. In particular, the class of functors having a right (left) adjoint is closed under composition. In the context of complete lattices, this is of course entirely trivial, being simply the affirmation that the composition of two join (meet)-preserving maps is itself join (meet)-preserving. We then recover the standard categories of complete join (meet) semilattices.

Finally, recall that an orthocomplementation on the lattice $(X, <)$ with minimal element 0 is an antitone involution $'$: $X \to X$ such that a' is a complement of a:

(1) $a'' = a$ for each $a \in X$;
(2) If $a < b$, then $b' < a'$;
(3) $a \wedge a' = 0$ for each $a \in X$.

In particular, if $F \dashv G$, then the map $\overline{G}$: $a_2 \mapsto (G(a'_2))'$ is join-preserving. We then recover the notion of the adjoint of a hemimorphism, introduced for orthomodular lattices by Foulis (1960) and developed in, for example, Gudder and Michel (1981), Piron (1995), Pool (1968a,b), and Rüttimann (1975).

4. CLOSURE OPERATORS

For posets, monads turn out to be nothing more than closure operators. For a given monad T the corresponding Eilenberg–Moore category is the poset of fixed points T, and the corresponding Kleisli category is the original set with the induced preorder $a < b$ if and only if $a < Tb$. Of particular importance is the case of atomistic lattices. In this context it is useful to consider simple closure operators, that is, closure operators which map each atom to itself. We then obtain equivalent categories of complete atomistic lattices and closure spaces. For a general discussion on the relationship between separation conditions on closure spaces and classes of lattices see Faure (1994).

Let $(X, <)$ be a poset. A closure operator on $(X, <)$ is a map $T: X \to X$ such that:

(1) If $a < b$, then $Ta < Tb$.
(2) $a < Ta$ for each $a \in X$.
(3) $TTa < Ta$ for each $a \in X$.

Note that $T \circ T = T$ for any closure operator. Indeed, $TTa < Ta$ by (3). However $a < Ta$ by (2), and so $Ta < TTa$ by (1).

Now it is intuitive to consider the elements $a \in X$ as subobjects of X by considering the canonical injections i_a: $[0, a] \to X$. Further, the notion of subobject can be adequately defined in any category X with respect to a distinguished class $\mathcal{M}$ of monomorphisms which is closed under composition and which contains all isomorphisms. Indeed, it suffices to define the class $\mathcal{M}/A$ of subobjects of A to be those monomorphisms in $\mathcal{M}$ with codomain A. The class $\mathcal{M}/A$ can then be preordered by setting

$$m < n \Leftrightarrow (\exists j)\ n \circ j = m$$

In this way one can define a local closure operator T_A for each object A of a given category. One can then define a global closure operator on the category itself by imposing continuity conditions to relate the different T_A (Cagliari and Cicchese, 1983; Dikranjan and Giuli, 1987); for a detailed exposition see Dikranjan and Tholen (1995).

Recall that a monad on the category $(X, <)$ is a functor $T: X \to X$ together with natural transformations η: Id $\to T$ and μ: $T \circ T \to T$, such that $\mu \circ T\mu = \mu \circ \mu T$ and $\mu \circ T\eta = \mu \circ \eta T = \mathrm{id}T$.

Lemma 4.1. Closure operators on the poset $(X, <)$ considered as a thin category are in bijective correspondence with monads.

Indeed, T is a functor, and so an isotone map. Further, the existence of a natural transformation η: Id $\to T$ implies that $a < Ta$. Finally, the existence of a natural transformation μ: $T \circ T \to T$ implies that $TTa < Ta$.

Recall that to any monad (T, η, μ) on the category X we can associate the Eilenberg–Moore category X^T, defined as follows. The objects are T-algebras, that is, pairs (a, α), where a is an object in X and α: $Ta \to a$ is a morphism such that $\alpha \circ \eta_a = \mathrm{id}_a$ and $\alpha \circ T\alpha = \alpha \circ \mu_a$.

Lemma 4.2. Let T be a closure operator on the poset $(X, <)$. Then the associated Eilenberg–Moore category X^T is the poset of fixed points of T with the induced partial order.

Indeed, there exists a morphism α: $Ta \to a$ if and only if $Ta < a$, and since T is a monad, we have that $a < Ta$. Hence T-algebras are exactly the fixed points of T. The partial order is the induced one, since all morphisms between T-algebras are in fact T-morphisms.

Recall that for any monad (T, η, μ) on the category X, the families of maps

$$U^T\colon \quad (a, \alpha) \mapsto a; f \mapsto f, \qquad F^T\colon \quad a \mapsto (Ta, \mu_a); \quad f \mapsto TF$$

are functors, with $F^T \dashv U^T$. Hence, F^T preserves colimits, and U^T preserves limits. Explicitly:

Lemma 4.3. Let T be a closure operator on the complete lattice $(X, <)$. Then the poset $(X^T, <)$ of fixed points of T is a complete lattice, with

$$\wedge_T \mathscr{A} = \wedge \mathscr{A}, \qquad \vee_T \mathscr{A} = T(\vee \mathscr{A})$$

Note that the fixed point lattice of the closure operator $T = R \circ L$ induced by an adjunction $L \dashv R$ is just the image of R, since $R \circ L \circ R = R$.

Recall that an atom of the lattice $(X, <)$ with minimal element 0 is a minimal nonzero element: $p \neq 0$; and if $a < p$, then either $a = 0$ or $a = p$. I write Σ_x for the (possibly empty) set of atoms of $(X, <)$. The lattice $(X, <)$ is called atomistic if each element is generated by its atoms:

$$a = \vee\{p \in \Sigma_x | p < a\} \qquad \forall a \in X$$

A closure operator T on the atomistic lattice $(X, <)$ is then called simple if

(1) $T0 = 0$;
(2) $Tp = p$ for each $p \in \Sigma_x$.

Lemma 4.4. Let T be a simple closure operator on the complete atomistic lattice $(X, <)$. Then $(X^T, <)$ is atomistic.

Indeed, the atoms of $(X^T, <)$ are exactly those of $(X, <)$, and if $Ta = a$, then

$$a = Ta = T(\vee\{p \in \Sigma_x | p < a\}) = \vee_T\{p \in \Sigma_x | p < a\}$$

A closure space is a set Σ together with a simple closure operator T on

$\mathcal{P}\Sigma$. Note that the atoms of $\mathcal{P}\Sigma$, and so of $(\mathcal{P}\Sigma)^T$, are exactly the singletons $\{p\}$ for $p \in \Sigma$. To each closure space (Σ, T) one can then associate the complete atomistic lattice $((\mathcal{P}\Sigma)^T, \subseteq)$. The converse is also true:

Lemma 4.5. Let $(X, <)$ be a complete atomistic lattice and define the maps

$$i_x\colon \quad X \mapsto \mathcal{P}\Sigma_x\colon \quad a \mapsto \{p \in \Sigma_x | p < a\}$$

$$\pi_x\colon \quad \mathcal{P}\Sigma_x \mapsto X\colon \quad A \mapsto \vee A$$

Then $\pi_x \dashv i_x$ with corresponding simple closure operator

$$i_x \circ \pi_x\colon \quad \mathcal{P}\Sigma_x \to \mathcal{P}\Sigma_x\colon \quad A \mapsto \{p \in \Sigma_x | p < \vee A\}$$

Indeed, i_x and π_x are both isotone, with

$$A = \{p \in \Sigma_x | p \in A\} \subseteq \{p \in \Sigma_x | p < \vee A\}$$

$$= \{p \in \Sigma_x | p < \pi_x(A)\} = (i_x \circ \pi_x)(A)$$

and

$$(\pi_x \circ i_x)(a) = \vee i_x(a) = \vee\{p \in \Sigma_x | p < a\} = a$$

It remains to prove that $i_x \circ \pi_x$ is simple:

$$(i_x \circ \pi_x)(0) = \{p \in \Sigma_x | p < \bigvee 0\} = \{p \in \Sigma_x | p < 0\} = 0$$

$$(i_x \circ \pi_x)(\{p\}) = \{q \in \Sigma_x | q \bigvee \{p\}\} = \{q \in \Sigma_x | q < p\} = \{p\}$$

Note that the elements of $(\mathcal{P}\Sigma_x)^{i_x \circ \pi_x}$ are exactly those $A \subseteq \Sigma_x$ for which there exists an $a \in X$ with $A = \{p \in \Sigma_x | p < a\}$.

5. CATEGORICAL EQUIVALENCES

Following the construction of Cl.-A. Faure and A. Frölicher, I now introduce equivalent categories of complete atomistic lattices and closure spaces based on the above object correspondence. I start by defining morphisms of complete atomistic lattices.

Lemma 5.1. Let $F\colon X_1 \to X_2$ and $G\colon X_2 \to X_1$ be functors between complete atomistic lattices such that $F \dashv G$. The following are equivalent:

(1) $F(\Sigma_{x1}) \subseteq \Sigma_{x2} \cup \{0_2\}$;
(2) $(\forall p_1 \in \Sigma_{x_1})(\exists p_2 \in \Sigma_{x_2})\ p_1 < Gp_2$.

In this case F will be called a morphism, and G a comorphism. We then obtain two dual categories, the equivalent conditions being preserved by composition. I now turn to morphisms of closure spaces.

Lemma 5.2. Let (Σ_1, T_1) and (Σ_2, T_2) be closure spaces, and f: $\Sigma_1\backslash K_1 \to \Sigma_2$ be a partially defined map. The following are equivalent:

(1) $f(T_1A_1\backslash K_1) \subseteq T_2f(A_1\backslash K_1)$ for each $A_1 \subseteq \Sigma_1$.
(2) If $T_2A_2 = A_2$ then $T_1\ (K_1 \cup f^{-1}(A_2)) = K_1 \cup f^{-1}(A_2)$.

If either, and so both, of these conditions are satisfied, then f is called a morphism. For f: $\Sigma_1\backslash K_1 \to \Sigma_2$ and g: $\Sigma_2\backslash K_2 \to \Sigma_3$, I define

$$g \circ f\colon \quad \Sigma_1\backslash K \to \Sigma_3\colon \quad p_1 \mapsto g(f(p_1)), \qquad K = K_1 \cup f^{-1}(K_2)$$

Note that $g \circ f$ is well defined, since if $p_1 \notin K$, then $p_1 \notin K_1$ and $f(p_1) \notin K_2$. We then obtain a category, the conditions being preserved by composition.

Having established a correspondence between objects, the next step is to establish a correspondence between morphisms. Let F: $X_1 \to X_2$ be a morphism of complete atomistic lattices, and define

$$f_F\colon \quad \Sigma_{x_1}\backslash K_1 \to \Sigma_{x_2}\colon \quad p_1 \mapsto F(p_1), \qquad K_1 = F^{-1}(0_2)$$

Note that f_F is well defined, since F maps atoms of Σ_{x_1} to either atoms of Σ_{x2} or 0_2.

Lemma 5.3. f_F is a morphism from $(\Sigma_{x_1}, i_{x_1} \circ \pi_{x_1})$ to $(\Sigma_{x_2}, i_{x_2} \circ \pi_{x_2})$.

On the other hand, let f: $\Sigma_1\backslash K_1 \to \Sigma_2$ be a morphism of closure spaces, and define

$$F_f\colon \quad (\mathcal{P}\Sigma_1)^{T_1} \to (\mathcal{P}\Sigma_2)^{T_2}\colon \quad A_1 \mapsto T_2f(A_1\backslash K_1)$$

$$G_f\colon \quad (\mathcal{P}\Sigma_2)^{T_2} \to (\mathcal{P}\Sigma_1)^{T_1}\colon \quad A_2 \mapsto K_1 \cup f^{-1}(A_2)$$

Lemma 5.4. We have the following results:

(1) F_f is a morphism of complete atomistic lattices.
(2) G_f is a comorphism of complete atomistic lattices.
(3) $F_f \dashv G_f$.

Finally, the above correspondences induce an adjunction. Let us define the families of maps

$$\mathbf{C}\colon \quad (X,<) \mapsto (\Sigma_x, i_x \circ \pi_x); \quad F \mapsto f_f, \qquad \mathbf{L}\colon \quad (\Sigma, T) \mapsto ((\mathcal{P}\Sigma)^T, \subseteq); \quad f \mapsto F_f$$

Lemma 5.5. We have the following results:

(1) **C** is a functor.
(2) **L** is a functor.
(3) **L** $\dashv$ **C.**

The two categories are then equivalent, since

$$\eta_\Sigma\colon\ \Sigma \to \mathbf{CL}\Sigma\colon\ p \mapsto \{p\}, \qquad \epsilon_x\colon\ \mathbf{LC}X \to X\colon\ A \mapsto \vee A$$

are isomorphisms, with respective inverses

$$\eta_\Sigma^{-1}\colon\ \mathbf{CL}\Sigma \to \Sigma\colon\ \{p\} \mapsto p, \qquad \epsilon_x^{-1}\colon\ X \to \mathbf{LC}\bar{X}\colon\ a \mapsto \{p|p < a\}$$

For example, an orthogonality relation on the set Σ is a symmetric and antireflexive relation which separates the elements of Σ:

(1) If $p \perp q$, then $q \perp p$.
(2) $p \not\perp p$ for any $p \in \Sigma$.
(3) If $p \neq q$, then there exists $r \in \Sigma$ such that $p \perp r$ and $q \not\perp r$.

Let $\perp$ be an orthogonality relation on the set Σ, and $A \subseteq \Sigma$. We define

$$A^\perp = \{q \in \Sigma | (\forall p \in A)\ q \perp p\}$$

The map $A \mapsto A^{\perp\perp}$ is then a closure operator, since the first condition implies that:

(1) If $A \subseteq B$, then $B^\perp \subseteq A^\perp$.
(2) $A \subseteq A^{\perp\perp}$ for each $A \subseteq \Sigma$.
(3) $A^{\perp\perp\perp} = A^\perp$ for each $A \subseteq \Sigma$.

Further, the second condition implies that the map $A \mapsto A^\perp$ is an orthocomplementation on the complete lattice $((\mathcal{P}\Sigma)^{\perp\perp}, \subseteq)$ of biorthogonal subsets of Σ, since

(4) $A \cap A^\perp = \emptyset$.

Finally, the third condition implies that each singleton is biorthogonal, so that $A \mapsto A^{\perp\perp}$ is a simple closure operator. In particular $((\mathcal{P}\Sigma)^{\perp\perp}, \subseteq)$ is atomistic.

To each orthogonal space we can then associate a complete atomistic orthocomplemented lattice. The converse is also true. Indeed, given an orthocomplementation $'$ on the complete atomistic lattice $(X, <)$ we define $p \perp q$ if and only if $p < q'$. In particular, we have that:

(1) $A^\perp = \{q \in \Sigma_x | q < (\vee A)'\}$ for each $A \subseteq \Sigma_x$.
(2) $\{p \in \Sigma_x | p < a\}^\perp = \{q \in \Sigma_x | q < a'\}$ for each $a \in X$.
(3) $A \subseteq \Sigma_x$ is biorthogonal if and only if $A = \{p \in \Sigma_x | p < \vee A\}$.

The categories of complete atomistic orthocomplemented lattices and orthogonal spaces are then equivalent.

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